# Electrodynamics of a two-electron atom with retardation and self-interaction

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We study the linear stability of a circular orbit in a two-electron atom, including retardation and self-interaction effects. We calculate all the eigenvalues of the linear stability problem, expanded up to third order in $v/c$. The retardation effects break the scale invariance of the Coulomb dynamics, and we discuss how this manifests in the linear stability of simple circular orbits. For some discrete energies, the linear eigenvalues define an extra resonant constant, which is important for the finite time stability of the orbit (e.g., emission of sharp spectral lines). We compare the magnitudes of the resonant orbits to the quantum atomic results. [S1063-651X(98)14811-0]

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## I. INTRODUCTION

The helium atom with Coulombian interaction is a prototype of a nonintegrable dynamical system. A simple property of the Coulomb interaction is its scale invariance; that is, the equations of motion are invariant under the scale transformation $t \rightarrow Tt$, $r \rightarrow Lr$ with $T^2/L^3 = 1$. Due to the finiteness of the speed of light, if one includes the retardation of the Coulomb interaction, the dynamical system is not scale invariant anymore. We have recently proposed that the retardation of the Coulomb interaction could stabilize some orbits of the helium atom by the existence of an extra resonant normal form constant [1]. This resonant constant could delay the ionization time of some discrete orbits up to a time scale consistent with the emission of a sharp spectral line.

In this paper we study in full the linear stability of a circular orbit of a two-electron atom with the inclusion of retardation and self-interaction. For the simple circular orbits, the stability analysis can be carried out analytically, which is not the case of generic chaotic orbits. A circular orbit is one where the two electrons are in the same circular orbit and in phase opposition, that is, along a diameter [2]. The center of the two-electron atom is supposed to have a positive charge of value $Ze$ $(Z > 1/4)$ and we henceforth call it the nucleus. The linearized dynamics about a circular orbit has one unstable direction, one stable direction, and ten neutrally stable directions [2]. For an infinitely massive nucleus, a two-electron atom has a six degree of freedom Hamiltonian system with only four independent constants of motion [3,4]. (Namely, these constants are the energy and the three components of the total angular momentum.) Because there are only four constants, the Coulombian dynamics in the neighborhood of a generic orbit can in principle be unstable (by lack of constants), and actually this dynamics is unstable about circular orbits [2]. Here we investigate if an extra resonant constant can exist in the neighborhood of a discrete set of circular orbits. This extra constant resonant normal form (adelphic integral [4]) requires a resonance to exist [4–8], as we discuss in Appendix D.

Historically, the understanding of the electrodynamics of a charged particle interacting with its own electromagnetic field [9–13] came very late. A classical solution to the self-interaction of a charged particle of very small radius is described by the Lorentz-Dirac equation of motion, henceforth called LDE. This equation was first obtained in 1938 by Dirac [10], using a covariant derivation without mention to the structure of the particle. Dirac was also the first to recognize and understand the runaway solutions of the LDE [13]. In this work we consider the classical electrodynamics of pointlike charges with a renormalized mass, as described by the Lorentz-Dirac theory [11–13].

Another late development of Maxwell's theory was the work of Page (1918) [14] on the expansion of the Liénard (1898) and Wiechert (1900) formula. This formula is complicated because of the retardation constraint, and one way to convert it into a useful differential equation is to develop the constraint in a Taylor series. This was done by Page up to the fifth order in 1918 [14], who also explored this formula in connection with self-interaction. We henceforth call the expansion of the Liénard-Wiechert interaction the "Page series." Truncated to second order in $v/c$, the Page series describes a Lagrangian interaction. This Lagrangian is the Darwin Lagrangian [13,15–17], which introduces the first retardation correction to the Coulomb dynamics. The Darwin Lagrangian is used in quantum mechanics to produce the Breit operator, which is the generalization of Dirac's relativistic wave equation for two-electron atoms, correct to second order in $v/c$ [18,19]. The Breit operator has been used successfully to describe the spin-orbit coupling and fine structure of helium [19,20]. The third-order term of the Page series is dissipative, and of course not Lagrangian.

The most studied case of a two-electron atom is the helium atom. The Coulomb dynamics of helium is not integrable, because only three constants of motion in involution exist for the dynamics [3,21]. This nonintegrability of helium appeared historically as a hindrance for the early quantization attempts of the Copenhagen school [22]. Quite recently, because of the renewed interest in periodic orbit quantization, some few numerical studies already exist on the Coulomb dynamics of helium [23,24]. For example, it is now known that most orbits lead to the self-ionization (ejection of one electron), after a long-term chaotic transient, for most initial conditions [24].

The introduction of the first retardation correction of the

 

Coulomb interaction modifies the dynamics in a quantitative and in a qualitative way, as far as this work is concerned: The Darwin Lagrangian is rotationally invariant, which generates an angular-momentum-like constant of motion according to Noether's theorem [21]. This constant is the angular momentum of the electrons, with relativistic correction, plus the angular momentum of the electromagnetic field. There is also an energylike constant (because the Darwin Lagrangian is time independent). These are small perturbations of the four constants of the Coulomb dynamics, and this is the quantitative change. The qualitative change is the existence of the extra resonant constant in the neighborhood of some orbits, and we stress that this is a genuine nonlinear effect, because the frequencies of the linearized dynamics with retardation depend nonlinearly on the orbit's frequency. This extra constant appears only after one includes the retardation effects, which unfold the scale invariance degeneracy of the Coulomb dynamics. In this sense, we have included the retardation because it is absolutely necessary, and not to attain some better precision.

In this paper we solve exactly a simple problem and we then apply the results heuristically to understand the resonance structure of complicated orbits. The exact simple problem is the complete stability analysis of a generic circular orbit, including retardation and self-interaction up to third order in $v/c$, which is done here in full detail for future reference for a generic two-electron atom. We develop a systematic method to handle the variational equations with inclusion of the retardation and self-interaction corrections. As a heuristic application of the linear problem we just solved, we explore some of the very interesting resonant orbits about which a resonant constant is possible. We show that a degeneracy determines which eigenvalue enters the resonance condition to produce orbits in the correct atomic magnitude. A few numerical experiments have convinced us that only finite perturbations of circular orbits can be stable for long time scales. In view of this, our stability analysis can only be used as an approximation to the stability of these more complicated chaotic orbits. Even though it is a mathematically sound idea that a small resonant perturbation can radically change the topology of a long-lived orbit [25], we cannot calculate analytically the resonance structure of such complicated orbits. Further numerical and analytical work should be done to check the heuristic criterion provided by the resonance condition and to obtain exact numerical quantities such as frequency of sharp lines, etc. The comparison of the magnitudes of the resonant orbits to the quantum results produces some astonishing agreement. This comparison immediately exposes the relevance of our heuristic approach to atomic physics and suggests what further results should be sought by use of rigorous nonlinear dynamics. The first striking coincidence is that our dynamical orbits have energies around the correct atomic magnitudes and the emitted frequencies agree very well with some frequencies of the spectra of helium and $\mathrm{Li}^+$ ion. The essential novelty in the work is that we are discussing a dynamical system that can emit a sharp line. (The other known case of emission of sharp line is the simple linear system [26]. To emit a sharp line, the orbit must be stable in a time scale of some $10^6$ turns, the type of stability that resonant normal form can provide.)

Some cautionary notes are in place before one starts reading this paper: First, we are studying dynamics in the neighborhood of a circular orbit, which is also a periodic orbit. One should not think though that our approach has anything to do with periodic orbit quantization [3]. It is known that in general EBK cannot be applied to generic orbits of Coulombian helium because they are unstable [3]. Second, the material in Sec. VI is heuristic or at the best an approximation aimed at introducing a few essentials of a brave new dynamical system.

This paper is organized as follows: In Sec. II we discuss the electromagnetic formulas. In Sec. III we consider the Coulomb stability problem, which is order zero of the Page series formula. In Sec. IV, we discuss the inclusion of the second-order terms and consider the necessary condition for the resonant constant. In Sec. V we consider the influence of the radiative terms, in Sec. VI we compare our results to the atomic results and in Sec. VII we put the discussions. In Appendix D we discuss the construction of the resonant normal form in an intuitive way for completeness.

## II. ELECTROMAGNETIC FORMULAS

In this work we include the self-interaction effects as described by the relativistic Lorentz-Dirac equation (LDE) with a renormalized mass [10–13]. The LDE equation for an electron of charge $-e$ can be written in the convenient, noncovariant form as

$$\frac{d}{dt}(\gamma M_e \dot{\mathbf{x}}_e) = \Gamma + \mathbf{F}_{\mathrm{ext}}, \tag{1}$$

where $\dot{\mathbf{x}}_e$ is the electron velocity, $M_e$ is the renormalized electronic mass [11], and $\gamma \equiv 1/\sqrt{1-(|\dot{\mathbf{x}}_e|/c)^2}$. In Eq. (1), $\mathbf{F}_{\mathrm{ext}}$ is the external force acting on the electron and $\Gamma$ is the radiation reaction force. For circular orbits, the lowest order term of $\Gamma$ in powers of $v/c$ is $\Gamma = (2e^2/3c^3)\dddot{\mathbf{x}}_e$. The next correction to $\Gamma$ is of order $(v/c)^5$ for a circular orbit (see, for example, page 116 of Ref. [12] for an expansion).

We now introduce, for later use, the expansion of the retardation constraint of the Liénard-Wiechert interaction (the Page series [14]). Let $\mathbf{x}$ be the position of a charge $q$, and $\boldsymbol{\beta}$ its velocity vector divided by $c$. The formula for the electric field caused by this charge $q$ at a point $p$ is

$$\mathbf{E}_p = q\frac{\hat{\mathbf{n}}}{r^2} + q\left\{\frac{[|\boldsymbol{\beta}|^2 - 3(\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2]\hat{\mathbf{n}}}{2r^2} - \frac{\dot{\boldsymbol{\beta}}}{2rc} - \frac{(\hat{\mathbf{n}}\cdot\dot{\boldsymbol{\beta}})\hat{\mathbf{n}}}{2rc}\right\} + \frac{2q}{3c^3}\dddot{\mathbf{x}} + \cdots, \tag{2}$$

where $\hat{\mathbf{n}}$ is the unit vector pointing from the charge to the point $p$, $r$ is the distance from the charge to $p$, and the ellipsis represents terms of order higher than 3 in $v/c$. Notice that all functions are evaluated at the present time. For a circular orbit, the term of Eq. (2) inside braces is of order $(v/c)^2$ times the Coulomb term and the term with the third derivative of position is of order $(v/c)^3$ times the Coulomb force.

A detailed expansion, correct to fifth order in $v/c$, is calculated in [14]. The magnetic field caused by the charge $q$ at $p$ has the following series:

$$\mathbf{B}=\frac{q}{r^2}[\boldsymbol{\beta}\times\hat{\mathbf{n}}]+\cdots. \tag{3}$$

This first term is the Biot-Savart term and the next term in the series would produce a force of fourth order in $v/c$ and along the normal, and it is not important for the present work. In this work we consider only the above terms of the electromagnetic interaction, and also consider the relativistic correction of Newton's law for the electronic motion up to second order in $v/c$.

Let us now discuss electrodynamics with retardation and self-interaction in the special case of the helium atom: We recall that the circular orbit is defined as one in which the two electrons are in the same circular orbit but 180 degrees out of phase [2]. Along such an orbit, the total force acting on electron 2 can be calculated using Eq. (2) and the self-interaction of electron 2 to be

$$\mathbf{F}=\frac{2e^2}{3c^3}(\dddot{\mathbf{x}}_1+\dddot{\mathbf{x}}_2)-7\frac{e^2\hat{\mathbf{n}}}{4R^2}(1-\tfrac{3}{7}|\beta|^2), \tag{4}$$

where $R$ is the radius of the circular orbit. In Eq. (4) we have also added the Coulomb attraction of the $\alpha$ particle and used the approximation that the $\alpha$ particle is infinitely massive and resting at the origin. If the two electrons are in the same circular orbit but in phase opposition, the force along the velocity cancels out [first term on the right of Eq. (4)], demonstrating that the circular orbit is a possible periodic solution of the electromagnetic equations up to third order.

Notice the appearance of the dipole term in Eq. (4),

$$\ddot{\mathbf{D}}=-e(\ddot{\mathbf{x}}_1+\ddot{\mathbf{x}}_2). \tag{5}$$

The total far field caused by the three particles depends linearly on the quantity $\ddot{\mathbf{D}}$, defined by Eq. (5), up to quadrupole terms. If this quantity is zero, the orbit is not radiating in dipole. The fifth-order terms of the Page series force and the fifth-order relativistic correction to the Lorentz-Dirac self-interaction introduce quadrupole effects. These effects would be important only in a much longer time scale, of order $T/(v/c)^5$. In Sec. VI we show that the circular orbits decay in a time of the order of $T/(v/c)^3$. Therefore, the effect of quadrupole terms is small during the orbit's lifetime. We start the next paragraph by discussing this importance of quadrupole terms in detail.

Throughout this paper, it is very important to keep in mind the orders of magnitude relevant to atomic physics. For example, a typical value for $v/c$ is $v/c\sim10^{-2}$. Typical values for the width of the spectral lines is of the order of $(v/c)^3/T$, which is the inverse of a time to perform about $10^6$ turns along a typical orbit [27]. In the classical model of the isolated hydrogen atom, because of dipole radiation losses, the energy loss during this linewidth time produces dramatic changes in the frequency of rotation and therefore a band of dipole radiation is emitted, not a sharp line, as is discussed in [26]. This is Bohr's [28] classical argument against sharp lines in the one-electron system of the isolated hydrogen atom. The argument obviously does not apply for two-electron systems like the helium atom or the hydrogen molecule, since those can have orbits with a zero dipole moment. For oscillations about circular orbits of helium, because there is no dipole radiation, a sharp line can be emitted, provided the orbit is otherwise stable. The quadrupole power radiated is of size $(E_0/T)(v/c)^6$, where $E_0$ is the Coulombian energy of the orbit and $T$ its period. This power times the linewidth time $T/(v/c)^3$ results in an energy loss of $E_0(v/c)^3$, which is consistent with a sharp variation of the emitted frequency.

A word of caution should be said about the fact that terms in the Page series of order higher than two represent a singular perturbation. Therefore, when one such term is included in the equations of motion, there will be solutions to the equations that are not a perturbation of a mechanical Coulombian orbit. We call these solutions ''nonmechanical,'' as opposed to ''quasimechanical'' regular perturbations. In this work we investigate only quasimechanical regular perturbations of circular orbits. To state it clearly, we operate under the conjecture that there is always a nonrunaway solution of the LDE in the neighborhood of conservative orbits [29].

If one wants the most generic ''stationary state,'' where the helium atom does not radiate in dipole, then one must have $\ddot{\mathbf{D}}=-e(\ddot{\mathbf{x}}_1+\ddot{\mathbf{x}}_2)=\mathbf{0}$ for all times. If we integrate the condition $\ddot{\mathbf{x}}_1+\ddot{\mathbf{x}}_2=\mathbf{0}$ twice in time, we get

$$x_1+x_2=a+bt. \tag{6}$$

The constant $b$ must be zero if the electrons are bound to the center of force at the origin. One can show that $a$ must also be zero as follows: Substitution of $x_1+x_2=a$ and $\ddot{\mathbf{x}}_1+\ddot{\mathbf{x}}_2=\mathbf{0}$ into the Coulombian equations of motion gives a polynomial equation for $a$, and $a=0$ is the only solution to this polynomial equation. Along orbits that satisfy Eq. (6) with $a=b=0$, the interaction between the electrons just renormalizes the charge of the $\alpha$ particle. The most general ''quasimechanical'' stationary orbit possible is then an elliptical orbit, with circular orbits as a particular case. In this work we do not consider elliptical orbits, which pose a more complicated parametric stability problem. The other kind of possible stationary orbits are symmetric collinear motions of the two electrons. These are orbits of zero angular momentum, which are also singular orbits (Coulombian solutions with zero angular momentum and zero dipole would fall onto the nucleus). We discuss them briefly in Sec. VII.

If one is not interested in nonmechanical orbits, it is convenient to truncate the Page series to third order and assume that the truncated system describes all the essential electrodynamics and that the next terms only introduce a small stochasticity. For nonmechanical orbits, such as zero angular momentum orbits, the Page series is not convergent and it might be necessary to keep all orders, like in Eliezer's theorem [26,30,31]. In situations where the Page series is at least asymptotic, the successive orders become important in successively longer time scales. For example, about a circular orbit of period $T$, the second order terms produce deviations from the Coulomb dynamics in a time of order $T/(v/c)^2$. The third-order terms take a time $T/(v/c)^3$ to influence the dynamics, and so on. In this work we consider the well-defined dynamical system obtained by truncating the Page series interaction to third order, and we include the self-interaction to third order in $v/c$ as well.

## III. COULOMBIAN STABILITY OF CIRCULAR ORBITS

In this section we consider the nonrelativistic Coulomb dynamics of a two-electron atom in the neighborhood of a circular orbit. The plain Coulomb stability problem has already been considered by many authors [2,32] using other approaches, and we do it our way in Appendix A to introduce our perturbation scheme. The Coulomb interaction is order zero of the Page series (2), and the scheme we develop allows for the inclusion of higher-order terms of the interaction in an easy way. An alternative equivalent way to perform this calculation is to transform to a coordinate system rotating with the frequency of the circular orbit. In this system the circular orbit is a fixed point of an autonomous vector field, the Jacobian matrix is independent of time, and the parametric problem is replaced by a linear eigenvalue problem. The disadvantage is that one has to transform all the terms of the Page series to the rotating coordinates. We get back to rotating coordinates later on.

Newton's equations of motion for the Coulombian two-electron atom are

$$M_\alpha \ddot{\mathbf{x}}_\alpha = \frac{Ze^2}{R_{1\alpha}^3}(\mathbf{x}_1 - \mathbf{x}_\alpha) + \frac{Ze^2}{R_{2\alpha}^3}(\mathbf{x}_2 - \mathbf{x}_\alpha),$$

$$M_e \ddot{\mathbf{x}}_1 = -\frac{Ze^2}{R_{1\alpha}^3}(\mathbf{x}_1 - \mathbf{x}_\alpha) - \frac{e^2}{R_{12}^3}(\mathbf{x}_2 - \mathbf{x}_1), \tag{7}$$

$$M_e \ddot{\mathbf{x}}_2 = -\frac{Ze^2}{R_{2\alpha}^3}(\mathbf{x}_2 - \mathbf{x}_\alpha) - \frac{e^2}{R_{12}^3}(\mathbf{x}_1 - \mathbf{x}_2),$$

where $\mathbf{x}_\alpha$, $\mathbf{x}_1$, and $\mathbf{x}_2$ are the position vectors respectively of the nucleus, electron 1 and electron 2, $R_{12} \equiv |\mathbf{x}_1 - \mathbf{x}_2|$, $R_{1\alpha} \equiv |\mathbf{x}_1 - \mathbf{x}_\alpha|$, and $R_{2\alpha} \equiv |\mathbf{x}_2 - \mathbf{x}_\alpha|$. In the special case of helium, the nucleus is an $\alpha$ particle, but we will use the index $\alpha$ to label the coordinate of the nucleus in the generic case as well. The nucleus has an arbitrary charge of $Ze$, and the electrons have charge $-e$. The circular orbit periodic solution of Eq. (7) is

$$x_\alpha = 0, \quad y_\alpha = 0, \quad z_\alpha = 0,$$

$$x_1 = R\cos(\omega t), \quad y_1 = R\sin(\omega t), \quad z_1 = 0, \tag{8}$$

$$x_2 = -R\cos(\omega t), \quad y_2 = -R\sin(\omega t), \quad z_2 = 0.$$

According to Eq. (7), the frequency of the orbit is related to $R$ by

$$M_e \omega_0^2 = \left(Z - \frac{1}{4}\right)\frac{e^2}{R^3}. \tag{9}$$

In this section, $\omega$ and $\omega_0$ are the same, but we will see later on that because of higher-order corrections, $\omega_0$ is only the first term of $\omega$ in powers of $(v/c)^2$. To simplify the notation we define the quantity

$$\phi \equiv \left(\frac{1}{8Z-2}\right),$$

which specifies a generic two-electron atom.

Linearizing Eq. (7) about the circular orbit (8), we obtain a parametric linear differential equation with coefficients periodic in time and period $T = \pi/\omega$, as we discuss in Appendix B. The study of this time-dependent linear system follows standard Floquet analysis, as described in Appendix A. It turns out, because of the symmetry of simple circular orbits, that the linear eigenvalue problem involves only six Floquet components of the linearization.

The resulting matrix equations are

$$\begin{bmatrix} -\left(\frac{1}{2} + 4\bar{n}^2\right) & \frac{-3i}{2} \\ \frac{3i}{2} & -\left(\frac{1}{2} + 4\bar{n}_+^2\right) \end{bmatrix} \begin{bmatrix} \xi_n^d \\ \chi_{n+1}^d \end{bmatrix} = 0, \tag{10}$$

and

$$4\begin{bmatrix} -\bar{n}^2 & 0 & Z\phi y & 3iZ\phi \\ 0 & -\bar{n}_+^2 & -3iZ\phi & Z\phi y \\ 0 & 0 & -(\bar{n}^2 + \Delta) & -3i\Delta \\ 0 & 0 & 3i\Delta & -(\bar{n}_+^2 + \Delta) \end{bmatrix} \begin{bmatrix} \xi_n^\alpha \\ \chi_{n+1}^\alpha \\ \xi_n^r \\ \chi_{n+1}^r \end{bmatrix} = 0, \tag{11}$$

where $\bar{n}_+ \equiv \bar{n} + 1$, $y \equiv (1/\varrho)$, and $\Delta \equiv Z\phi(1+2y)$. Once $\bar{n}$ is calculated from the secular condition, we can recover the Floquet multiplier $\mu$ because any complex number can be decomposed in a unique way as $\bar{n} = n + \mu$ with $n$ an integer and $|\mathrm{Re}(\mu)| < \frac{1}{2}$. As explained in Appendix A, the six linear equations separate in blocks of two and four when we introduce the convenient radiation and difference coordinates. At this point the reader should take a look in Appendix A to see the definition of the six Floquet components appearing in the equations. (The $2\times 2$ block depends only on the Floquet transform of the difference coordinates and the $4\times 4$ block depends only on the Floquet transforms of the radiation and $\alpha$ particle coordinates.) Notice that the secular problems of (10) or (11) involve only $\bar{n}$ and $\bar{n}_+ \equiv \bar{n} + 1$. After we find $n$, this defines that the only two nonzero components are the $n$th and $(n+1)$th in Eq. (A1) for the normal mode oscillation. An eigenvector calculation should then follow to determine the ratio of these two Floquet components.

Finally, let us solve Eqs. (10) and (11) for the roots $\bar{n}$. For Eq. (10) the roots are

$$\bar{n} = -1, 0, -1/2, -1/2.$$

As regards the eight roots of Eq. (11), four roots are 0,0, $-1,-1$. From the general theory of linear ODE's, at a double root like $\bar{n} = 0$, the general solution is a ''quasipolynomial'' linear function of time [36]. The doubly degenerate root $\bar{n} = 0$ is then responsible for the homogeneous translation solution with a constant velocity of the Coulombian helium. Evaluating the determinant of Eq. (11) and equating it to zero, one finds that the other roots are given by

$$\overline{n} = -\tfrac{1}{2} + \epsilon,$$

where $\epsilon$ is a solution of

$$\epsilon^4 + \left(2\Delta - \frac{1}{2}\right)\epsilon^2 + \left(\frac{1}{4} - 2\Delta\right)\left(\frac{1}{4} + 4\Delta\right) = 0, \tag{12}$$

which is a quartic equation with solutions

$$\epsilon^2 = \tfrac{1}{4} - \Delta \pm \sqrt{\Delta(9\Delta - 1)}.$$

Inspection of the above formula shows that there is always a pair of roots with negative $\epsilon^2$ and a real pair of roots. The pair with negative $\epsilon^2$ describes an instability, which was first found by Nicholson [32] and in this work we refer to it as the Coulombian radial instability [2,24]. This is an exponential growth in a time scale of the order of one cycle. The corresponding roots are given by

$$\overline{n} = -\tfrac{1}{2} \pm i\sqrt{\Delta - \tfrac{1}{4} + \sqrt{\Delta(9\Delta - 1)}}, \tag{13}$$

where $i \equiv \sqrt{-1}$. One also finds that $\delta \mathbf{x}_r$ is not zero for the radially unstable mode, which implies that perturbations along this unstable mode radiate in dipole. We discuss some consequences of this in Sec. V. The other two roots of Eq. (12), with positive $\epsilon^2$ are approximated by

$$\overline{n} = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \Delta + \sqrt{\Delta(9\Delta - 1)}}. \tag{14}$$

Notice that the quantity $\zeta$ that appears in the paper of Poirier [2] is equal to 8 times our $\Delta$ $[\Delta \equiv Z(1+2y)/(8Z-2)]$ and that we are including the dynamics of the nucleus as well. To recover the results of Poirier one should put $y=0$ in our definition of $\Delta$.

Last, we consider the oscillations perpendicular to the plane of the orbit, which we call the $z$ direction. In linear order in the oscillation amplitude, this oscillation is decoupled from the oscillations along the plane. Starting from Eq. (7), and linearizing about Eq. (8) we obtain

$$\frac{\delta \ddot{z}_d}{\omega_0^2} + \delta z_d = 0,$$

$$\frac{\delta \ddot{z}_\alpha}{\omega_0^2} - (8Z\phi y)\,\delta z^r = 0, \tag{15}$$

$$\frac{\delta \ddot{z}_r}{\omega_0^2} + (8\Delta)\,\delta z_r = 0.$$

The solutions to this linear problem are of type

$$\delta z_\kappa = c_\kappa \exp(2i\omega\lambda). \tag{16}$$

The first equation is a simple separate linear equation and the solution to it with $c_d \neq 0$ requires $\lambda = \pm 1/2$. The next two equations of Eq. (15) become

$$\begin{bmatrix} -4\lambda^2 & -8Z\phi y \\ 0 & -4\lambda^2 + 8\Delta \end{bmatrix}\begin{bmatrix} c_\alpha \\ c_r \end{bmatrix} = 0, \tag{17}$$

and a nontrivial solution requires $\lambda = 0,0$ and

$$\lambda = \pm\sqrt{2\Delta}. \tag{18}$$

The linear $z$ oscillation is decoupled from the planar oscillation, but of course it couples at higher orders in the oscillation amplitudes.

As we mentioned in the beginning of this section, in a frame rotating with the frequency of the circular orbit, the circular orbit itself is a fixed point of an autonomous vector field. This vector field describes the Coulomb dynamics in the rotating frame. One finds that the inclusion of the second order and radiative terms still yields an autonomous vector field for the dynamics in the rotating frame. This is the reason why we were able to simplify the parametric equation down to a $6\times 6$ linear system. In this rotating system one also has to diagonalize a $6\times 6$ matrix (two plane coordinates for each particle), and the exact same problem along the $z$ direction.

## IV. INCLUSION OF SECOND-ORDER TERMS

In this section we consider the inclusion of the second-order terms of the Page series and the second-order relativistic corrections to the electronic dynamics. It is known that the Liénard-Wiechert interaction truncated to second order in $v/c$ is described by the Darwin Lagrangian [13,15]

$$L_{\text{Darwin}} = \sum_i \left(\frac{1}{2} m_i |\dot{\mathbf{x}}_i|^2 + \frac{1}{8c^2} m_i |\dot{\mathbf{x}}_i|^4\right) - \frac{1}{2}\sum_{ij} \frac{q_i q_j}{r_{ij}}\left(1 - \frac{1}{2c^2}[\dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_j + (\dot{\mathbf{x}}_i \cdot \hat{\mathbf{n}}_{ij})(\dot{\mathbf{x}}_j \cdot \hat{\mathbf{n}}_{ij})]\right), \tag{19}$$

where the indices take the values 1, 2, and $\alpha$, $r_{ij} \equiv |\mathbf{x}_i - \mathbf{x}_j|$, and $\hat{\mathbf{n}}_{ij}$ is the unit vector in the direction of $\mathbf{x}_i - \mathbf{x}_j$. This Lagrangian depends only on the scalar product of the velocities and the distance between the particles, therefore it is invariant under a global rotation of all the particle's coordinates. By Noether's theorem [21], this generates an angular-momentum-like constant of the motion associated with the symmetry, which is equal to the mechanical angular momentum plus a small functional correction of order $(v/c)^2$, representing the angular momentum of the electromagnetic field. Because the Lagrangian is time independent, there is also an energy constant of the motion. This makes four independent constants of the motion. According to [8], for an extra (formal) analytic constant to exist, some extra resonance condition must be satisfied by the linear frequencies. This extra constant is likely to be defined by a divergent series, which upon optimal truncation might provide an extra quasiconstant for a finite time scale. In the following we study the second-order correction of the Coulomb eigenvalues.

Let us start by correcting the frequency of the orbit as given by Eq. (9). A given circular orbit is characterized by the velocity $|\beta|$, and from this one can calculate all the other quantities of the orbit: angular momentum, frequency, and radius. The Page series gives a correction in powers of $|\beta|$, and the first correction is of order $|\beta|^2$, as it should be for any relativistically invariant dynamics [17]. Adding up the second-order magnetic (3) and second-order electric (2) forces acting on the electron along the circular orbit we find the normal force

$$\mathbf{F}=-\frac{e^2\mathbf{n}}{8\phi R^2}-\frac{|\beta_e|^2e^2\mathbf{n}}{8R^2}.$$

Together with the second-order relativistic mass correction, this determines the second-order correction to the frequency of Eq. (9) to be

$$\omega^2=\omega_0^2[1+(\phi-\tfrac{1}{2})|\beta|^2].$$

A simple way to include this second-order frequency correction is to multiply $\overline{n}$ and $\overline{n}_+$ by $[1+(\phi-\frac{1}{2})|\beta|^2]$ on the zero-order matrices of Eqs. (10) and (11) before adding the other second-order matrices. At this point one should look up Appendix B, where we evaluate the variation of the second-order terms of the Page series, as well as second-order relativistic corrections.

Next we evaluate the second-order corrections of the eigenvalues. The algebraic manipulations were performed using a symbolic manipulator program (MAPLE version 4.0). Adding up the second-order electric field, second-order magnetic force, the second-order relatistic correction for the electronic masses and the correction to the frequency, we obtain a perturbation to the matrices (10) and (11). We write the equation for the Floquet components, which separates in two parts just like Eq. (A4) splits into Eqs. (10) and (11). The resulting second-order correction to be added to Eq. (10) is

$$|\beta|^2\begin{bmatrix} P_2(\overline{n}) & iQ_2(\overline{n}) \\ -iQ_2(-\overline{n}_+) & P_2(-\overline{n}_+)\end{bmatrix}\begin{bmatrix}\xi_n^d \\ \chi_{n+1}^d\end{bmatrix}, \tag{20}$$

where $P_2(\overline{n})\equiv[(8\phi-2)\overline{n^2}+8\phi\overline{n}+3\phi/2]$, $Q_2(\overline{n})\equiv[(4\phi+2)(\overline{n^2}-\overline{n})+\phi/2]$, and $\overline{n}_+\equiv(1+\overline{n})$. Let us first consider the correction in powers of $|\beta|$ of the roots of Eq. (10). This is done by adding up matrices (10) and (20), and taking the determinant of the result. The result, up to second order in $|\beta|$, is

$$\begin{aligned}&16\overline{n}(1+\overline{n})(1+2\overline{n})^2+|\beta|^2\{(7+6\phi)+8(1-5\phi)\overline{n}\\&\quad+8(3-13\phi)\overline{n^2}+16(1-4\phi)(2\overline{n^3}+\overline{n^4})\}+\cdots.\end{aligned} \tag{21}$$

In the neighborhood of the simple root $\overline{n}=0$ of (10), Eq. (21) takes the form

$$16\overline{n}+(7+6\phi)|\beta|^2=0,$$

which yields $\overline{n}=-(7+6\phi)|\beta|^2/16$. Analogously, in the neighborhood of $\overline{n}=-1$, Eq. (21) takes the form

$$-16(\overline{n}+1)+(7+6\phi)|\beta|^2=0,$$

which yields $\overline{n}=-1+(7+6\phi)|\beta|^2/16$. Last, and crucial for this work, is the bifurcation of the $\overline{n}=-1/2$ double root. About $n=-1/2$, Eq. (21) becomes

$$-4(1+2\overline{n})^2+6(1+2\phi)|\beta|^2=0,$$

with roots

$$\overline{n}=-\frac{1}{2}\pm|\beta|\sqrt{\frac{3(1+2\phi)}{2}}. \tag{22}$$

Notice that the correction of the degenerate root in Eq. (22) comes with a *linear* power of $|\beta|$, differently from the simple roots, that are corrected only at order $|\beta|^2$ (exactly because of this degeneracy). This root undergoes the fastest change of all for small $v/c$, and one should expect it to be the first to accommodate a ''new'' resonance condition. In the construction of a resonant normal form constant, this frequency allows the resonance condition to be satisfied with the lowest possible value of $|\beta|$. (In the same way explained in Sec. VI, one finds by inspection that resonances among frequencies corrected only at second order lead to relativistic orbits, not very interesting for atomic physics.)

Next we calculate the second-order corrections of Eq. (11): Adding up all the second-order corrections to Eq. (11), second-order electric, second-order magnetic, second-order relativistic and correction of the frequency we find the following matrix to be added to Eq. (11)

$$|\beta|^2\begin{bmatrix}(2-4\phi)\overline{n^2} & 0 & 0 & 0\\ 0 & (2-4\phi)\overline{n_+^2} & 0 & 0\\ R(\overline{n}) & iS(\overline{n}) & T(\overline{n}) & iW(\overline{n})\\ -iS(-\overline{n}_+) & R(-\overline{n}_+) & -iW(-\overline{n}_+) & T(-\overline{n}_+)\end{bmatrix}\times\begin{bmatrix}\xi_n^\alpha\\ \chi_{n+1}^\alpha\\ \xi_n^r\\ \chi_{n+1}^r\end{bmatrix}, \tag{23}$$

where $R(\overline{n})\equiv-8(1+3\phi)\overline{n^2}+8\phi\overline{n}$, $S(\overline{n})\equiv4(1-\phi)\overline{n^2}+(2\phi-4)\overline{n}$, $T(\overline{n})\equiv-2(1+8\phi)\overline{n^2}+4\phi\overline{n}$, and $W(\overline{n})\equiv2(1-2\phi)\overline{n^2}-2\overline{n}$, and we have left out terms proportional to $y=1/\varrho$. To calculate the corrections to the roots of Eq. (11), we add Eq. (23) to Eq. (11), take the determinant and equate it to zero. Adding Eq. (23) to Eq. (11) and evaluating the determinant we obtain, up to second order in $|\beta|^2$,

$$\begin{aligned}&256\overline{n^2}(1+\overline{n})^2\Bigg\{(\overline{n^4}+2\overline{n^3}+(1+2\phi)\overline{n^2}+2Z\phi\overline{n}\\&+Z\phi-8Z^2\phi^2)+10\phi|\beta|^2\Bigg[\overline{n^4}+2\overline{n^3}+\left(\frac{9}{10}+\frac{Z}{5}+\frac{6Z\phi}{5}\right)\overline{n^2}\\&-\left(\frac{1}{10}+\frac{Z}{5}+\frac{5Z\phi}{3}\right)\overline{n}\\&+(-\tfrac{16}{5}Z^2\phi^2+\tfrac{8}{5}\phi Z^2+\tfrac{7}{5}Z\phi+\tfrac{1}{2}Z)\Bigg]\Bigg\}.\end{aligned} \tag{24}$$

TABLE I. Corrections to the sixteen stable regular roots, $\phi \equiv 1/(8Z-2)$, and $\Delta \equiv Z\phi(1+2y)$.

| Coordinate involved | Coulombian roots | Corrected roots |
|---|---|---|
| $\delta z^d$ | $\lambda = \pm \frac{1}{2}$ | $\lambda = \pm \frac{1}{2}\left[1-\left(\frac{2-5\phi}{4}\right)\lvert\beta\rvert^2\right]$ |
| $(\delta z^\alpha, \delta z^r)$ | $\lambda = 0,0,\pm\sqrt{2\Delta}$ | $\lambda = 0,0,\pm\sqrt{2\Delta}(1-\frac{3}{2}\phi\lvert\beta\rvert^2)+\frac{64i}{3}Z\phi^2\lvert\beta\rvert^3$ |
| $\delta \mathbf{x}^d$ | $\overline{n}=0,-1,-\frac{1}{2},-\frac{1}{2}$ | $\overline{n}=-\frac{7+6\phi}{16}\lvert\beta\rvert^2,-1+\frac{7+6\phi}{16}\lvert\beta\rvert^2,-\frac{1}{2}$ $\pm\sqrt{\frac{3+6\phi}{2}}\lvert\beta\rvert$ |
| $(\delta \mathbf{x}^\alpha, \delta \mathbf{x}^r)$ | $\overline{n}=0,0,-1,-1,$ $\frac{-1}{2}\pm\sqrt{\frac{1}{4}-\Delta+\sqrt{\Delta(9\Delta-1)}}$ | $\overline{n}=0,0,-1,-1,-\frac{1}{2}\pm\sqrt{\frac{1}{4}-\Delta+\sqrt{\Delta(9\Delta-1)}}$ $\pm c(Z)\lvert\beta\rvert^2+id(Z)\lvert\beta\rvert^3$ |

Notice that according to Eq. (23) we still have the degenerate roots at $\overline{n}=0$ and $\overline{n}=-1$. An inspection in Eq. (24) shows that from all the roots given by Eq. (11), only $\overline{n}=0$ and $\overline{n}=-1$ are still degenerate after the inclusion of the term in $|\beta|^2$. We will not develop it here, but for $y\neq 0$ these degenerate roots become simple roots and each other root 0 and $-1$ is corrected by terms proportional to $y|\beta|^2$. The roots of Eq. (14) are nondegenerate and are corrected only at second order as

$$\overline{n}=-\tfrac{1}{2}\pm\sqrt{\Delta-\tfrac{1}{4}+\sqrt{\Delta(9\Delta-1)}}\pm c(Z)|\beta|^2. \tag{25}$$

The function $c(Z)$ is a complicated function of $Z$, and we give its value for the most interesting values of $Z$, namely, $C(2)=0.15241$, $C(3)=0.11926$, and $C(4)=0.10126$.

As regards the second-order corrections for oscillations along the $z$ direction, the equation for $\delta z_d$ is changed to

$$\frac{\delta\ddot{z}_d}{\omega_0^2}+\delta z_d+|\beta|^2\left[\left(\tfrac{1}{2}-2\phi\right)\frac{\ddot{z}_d}{\omega_0^2}-\phi z_d\right]=0,$$

and the roots are changed to

$$\lambda=\pm\frac{1}{2}\left[1-\left(\frac{1}{2}-\frac{5\phi}{4}\right)|\beta|^2\right].$$

In an analogous way, we find that the second-order correction to Eq. (17) is

$$|\beta|^2\begin{bmatrix} 4\left(\frac{1}{2}-\phi\right)\lambda^2 & 0 \\ 16\left(2Z\phi-\frac{1}{4}-\phi\right) & -12\phi\lambda^2 \end{bmatrix}\begin{bmatrix} c_\alpha \\ c_r \end{bmatrix}. \tag{26}$$

Adding Eq. (26) to Eq. (17), taking the determinant, and equating it to zero, we find that the $\lambda=0$ double root is preserved and the roots of Eq. (18) are corrected to

$$\lambda=\pm\sqrt{2Z\phi}\left(1-\frac{3\phi}{2}|\beta|^2\right). \tag{27}$$

This completes the calculation of the second-order correction to all the Coulombian roots. In the next section we calculate the third-order corrections, and Table I shows all the roots corrected to third order.

## V. INCLUSION OF THE DISSIPATIVE THIRD-ORDER TERMS

The terms of order higher than two in Eq. (2) are *singular* in the sense that they introduce the third derivative, and bring up a new solution to the dynamics. The next natural step in the study of the stability of the circular orbit would be to include the third-order terms in the calculation of the eigenvalues. In the following we calculate the third-order corrections, in a way analogous to the second order. Starting with the third-order correction to Eq. (17), we find the third-order matrix

$$\left(\frac{256i\phi|\beta|^2}{3}\lambda^3\right)\begin{bmatrix} 0 & 0 \\ (2-Z) & 1 \end{bmatrix}\begin{bmatrix} c_\alpha \\ c_r \end{bmatrix},$$

which describes the radiative correction to be added to Eq. (17), in disregard of terms proportional to $y|\beta|^2$. Solving the perturbed secular determinant, we find that the $\lambda=0$ double root is preserved and the roots of Eq. (18) are corrected to

$$\lambda=\pm\sqrt{(2Z\phi)}\left[1-\tfrac{3}{2}\phi|\beta|^2\right]+\tfrac{64}{3}iZ\phi^2|\beta|^3, \tag{28}$$

where again $i$ stands for the complex unit $i\equiv\sqrt{-1}$.

Last, let us calculate the third-order corrections to Eqs. (10) and (11). Because of symmetry, one finds that there is no third-order correction to Eq. (10). This is readily seen because the third-order variational force is the same for both electrons, being proportional to the $\delta\mathbf{x}_r$ variation. The third-order correction to Eq. (11) can be calculated using the third order of Eq. (2), in an analogous way we used for the second order. We find the following matrix describing the third-order correction to Eq. (11):

$$\frac{128}{3} i\phi|\beta|^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2(2-Z)\overline{n^3} & 0 & 2\overline{n^3} & 0 \\ 2(2-Z)\overline{n^3_+} & 0 & 2\overline{n^3_+} & 0 \end{bmatrix} \begin{bmatrix} \xi_n^\alpha \\ \chi_{n+1}^\alpha \\ \xi_n^c \\ \chi_{n+1}^c \end{bmatrix}.$$

Adding the above matrix to Eq. (11), add the second-order equation (23), and take the determinant. The result produces a correction to Eq. (24) given by

$$\delta_3 = -\frac{16\,384|\beta|^3 i}{3}(1+\overline{n})^2\overline{n^2}(1+2\overline{n})(\overline{n^4}+2\overline{n^3}+\overline{n^2} + Z\phi\overline{n^2}+Z\phi\overline{n}+Z\phi). \tag{29}$$

From the above, it is easy to see that the eigenvalues $\overline{n}=-1/2$, $\overline{n}=-1$, and $\overline{n}=0$ do not acquire any imaginary part at third order. It is also straightforward to add Eq. (29) as a perturbation to Eq. (11) and calculate the imaginary correction to Eq. (14) (using the program MAPLE). The imaginary correction is of type $id(Z)$, and the most interesting values of $d$ are $d(2)=0.766$, $d(3)=0.740$, and $d(4)=0.731$. In Table I we show all the corrections of the eigenvalues, including order $|\beta|^3$ correction. We only showed in Table I the correction to the regular roots, by which we mean the ones that were already roots of the Coulomb dynamics. The introduction of the third order brings up nine more singular roots, which we do not consider. Last, we did not show in Table I the unstable pair of eigenvalues (13) of the radial instability either, which are part of the 18 Coulombian eigenvalues but are not interesting for resonance conditions. The fact that we left out the singular roots is consistent with the conjecture stated in Sec. II that there is always a conservative solution in the neighborhood of the exact nonrunaway solution of the LDE [29]. As far as the next heuristic section goes, the very large singular linear frequencies would predict resonant orbits only in the relativistic energy region, which is another reason to believe they are unphysical.

## VI. HEURISTIC CRITERION OF RESONANT ORBITS

In this section we explore the orders of magnitude of some of the orbits. Since the retardation of the Coulombian interaction breaks the scale invariance of the equations of motion, one expects to find discrete stable orbits at some particular order of magnitude, as a signature of this lack of scale invariance. We explore here the resonance condition as a heuristic tool to predict the order of magnitude of these stable orbits. Rigorous numerical integration should follow to prove this heuristic criterion. As a matter of fact, a few numerical experiments have convinced us that the most long-lived orbits are finite perturbations of elliptical orbits [24]. Even if infinitesimal perturbations of circular orbits are quickly ionized, the information derived from the tangent dynamics of those can approximate the tangent dynamics of more complex nonionizing chaotic orbits for which we cannot hope to have simple analytical formulas.

The section is designed to inspect some of the resonant orbits predicted by normal form theory using the information already at hand. The two guiding dynamical principles we use are as follows: First, resonances with the minimal integer multipliers are the most important. It is known in general [40] that the size of the resonance islands varies as exp $(-o)$, where $o$ is the order of the resonance. This solves the paradox as to the infinite number of possible resonances: the ones with a high order occupy an exponentially small area of phase space, which makes them very unlikely. The situation is analogous to quantum mechanics, where there is always an infinity of energy levels. Nevertheless, in practice only a very small finite number of lines are observed in laboratory experiments on earth. For example, in the Balmer series, only the first twelve frequencies of the series can be observed as emission lines in very diluted gaseous states [20,28]. Since very high quantum states are too extended in space, one needs very rarefied gases and large astronomical masses of gas to produce a measurable signal. We consider in this section only some of the resonances with minimal ordering. Second, we operate under the guiding heuristic principle that this extra complex constant, together with the other four constants of helium can stabilize the six degree of freedom dynamics of helium for a time scale consistent with the emission of a sharp spectral line.

An approximation that we use in examining the resonance conditions is that we only include the correction to the frequency that comes with linear order, because this is the largest correction for small $|\beta|$. Let us now consider the special case of helium: As we mentioned before, according to the Darwin Lagrangian, helium has always an angular momentumlike constant of motion and an energylike constant. In the neighborhood of some select circular orbits, another complex analytic constant might exist, which could make those orbits nonlinearly stable. The constant we find here involves the amplitude of the normal mode corresponding to Eq. (18) in a combination with normal modes along the plane. This is possible because these modes are coupled at higher orders in the oscillation amplitude. We do not want to involve the linear modes describing the circular instability in the first resonant term of the constant, but the amplitudes of the unstable mode will naturally appear in the higher monomials of the series for the constant (in agreement with the numerical knowledge that a long-lived orbit is a finite perturbation of a circular initial condition). Besides, the circular pair (13) are complex eigenvalues and would not satisfy a simple resonance condition in combination with the other real eigenvalues.

The circular orbit is a fixed point of the autonomous vector field describing the dynamics in a system rotating with the frequency of the circular orbit. To develop the normal form about the fixed point, we must move to this rotating frame. We recall that $\overline{n}$ is a frequency of oscillation, in units of $2\omega$, of the variational dynamics, according to Eq. (A1). In the rotating system, the new frequencies for oscillation along the plane are found by adding 1/2 to the formulas (14) and (22). The frequency (18) for the $z$ oscillation is unchanged. A new resonance condition involving the root (22) is the easiest to be satisfied for small values of $|\beta|$. Therefore, we suggest to look for a resonance among the frequencies of Eqs. (14), (18), and (22), which we rename as $\omega_1$, $\omega_2$, and $\omega_3$, respectively, in the rotating frame, and which in the case of helium evaluate to

$$\omega_1 \equiv \sqrt{\frac{2}{7}} \simeq 0.5345\cdots$$

$$\omega_2 \equiv \sqrt{\frac{3+\sqrt{32}}{28}} \simeq 0.5560\cdots \tag{30}$$

$$\omega_3 \equiv \sqrt{\frac{12}{7}}|\beta| \simeq 1.3093|\beta|,$$

in units of $2\omega$, and we have disregarded corrections proportional to $y$ and $|\beta|^2$. According to standard normal form theory, the necessary condition to have an additional analytic [6,8] constant of the motion in the neighborhood of a fixed point is a resonance among the frequencies. By inspection, we find that a new quartic resonance involving the above three frequencies and with the minimal integer multipliers is of type

$$\omega_1 - \omega_2 + 2\omega_3 = 0. \tag{31}$$

This resonance is satisfied for $|\beta|$ given by

$$|\beta| = 0.0082\cdots.$$

The Coulombian binding energy of a circular orbit in a two-electron atom (kinetic plus potential) can be written as $E = -mc^2|\beta|^2$. This energy, in atomic units, for the above value of $|\beta|$ is

$$E = -1.265 \ \text{a.u.}$$

Of course, other resonance conditions are possible: for example, we could put an integer number in the resonance condition, as in

$$\omega_1 - \omega_2 + 2n\omega_3 = 0, \tag{32}$$

and the values of $|\beta|$ satisfying this condition are given by

$$|\beta| = \frac{0.0082}{n}.$$

The discrete circular orbits corresponding to these values of $|\beta|$ have binding energies given by

$$E = -\frac{1.265}{n^2} \ \text{a.u.}$$

In the next section we show how to construct the extra complex constant of the motion in a Maclaurin series (resonant normal form) when condition Eq. (32) is satisfied.

The surprising fact that we were able to pick particular circular orbits is a genuine signature of the nonlinear dynamics, because the linear eigenvalues depend on the circular orbit through the parameter $|\beta|$. The frequencies of the resonant orbits satisfying Eq. (32) are given by

$$\omega = \frac{0.8101}{n^3} \ \text{a.u.},$$

and the frequency of the $z$ oscillation in the stable manifold of the orbit can be obtained by multiplying the above frequency by $2\sqrt{2/7}$, as of Eq. (18)

$$w_z = \frac{0.866}{n^3}. \tag{33}$$

Supposing that the dynamics in the neighborhood of the resonant orbits is stable, one could expect that oscillations about this orbit could emit a sharp line. The only condition to emit a sharp line is to oscillate with the same frequency for a long enough time (of the order of the inverse of the width of the line). This condition is fulfilled because of the finite time stability of the resonant orbit. Here we assume that the frequencies of Eq. (33) are an approximation to the sharp lines emitted by the orbit.

For the circular orbit corresponding to Eq. (33) with $n=1$, the frequency of the $z$ oscillation in the stable manifold is 0.7956 atomic units. The transition from the first excited state of parahelium to the ground state ($2^1P \rightarrow 1^1S$) corresponds to a frequency of $2.9037-2.1237=0.7799$ atomic units [33], which is a 9% difference. For $n=2$, Eq. (33) evaluates to 0.1083 a.u., and the frequency for the transition ($3^1P \rightarrow 2^1S$) in parahelium is $2.1459-2.0551=0.0908$ a.u., which is again a 9% difference. Last, the asymptotic form of the quantum energy levels of helium [19], both parahelium or orthohelium, is

$$E = \frac{-Z^2}{2} - \frac{(Z-1)^2}{2(n_r+L+1)^2},$$

with $Z=2$, which is a first approximation to the Rydberg-Ritz spectroscopic term [20]. The frequency of the line emitted by transitioning to the neighboring level, calculated from the above formula with $\Delta L=1$ is approximated by

$$w = \frac{1}{(n_r+L+1)^3}.$$

This above equation is a quantum formula, which we write just to compare with Eq. (33). Notice that Eq. (33) agrees with it to within 13%.

We can also produce an estimate for the width of the line as follows: The third-order correction to the frequency of Eq. (28) ($w_z$) is imaginary and with the same sign for the two values of $\lambda$. Therefore the $z$ oscillation is stable and decays with a coefficient that, according to Eq. (16), is the imaginary part of $2\omega\lambda$. For example, in the case of helium, $Z=2$, and $\phi=1/14$, this imaginary correction to Eq. (33) evaluates to

$$\nu = i\frac{64}{49}|\beta|^3\omega,$$

and this oscillation is then part of the stable manifold of the circular orbit. Along this decaying oscillation, the perturbed circular orbit decays back to the perfectly circular special circular orbit. The radiative self-interaction has already been shown to produce good approximations for the linewidth in other situations [27]. If we evaluate the above result for the line at $\omega=0.7959$ a.u., we find that it is 1.5 times the experimental linewidth for this transition [33,34]. We stress again that this is not exactly the probability of decay of the orbit: As we already mentioned, the resonance condition (31) is not satisfied with the inclusion of the third-order terms, which implies that the constant of motion is destroyed. Because of this, the perturbed orbit can decay not only back to the perfectly circular circular orbit, but also to the lower-energy ground state. The dynamical linewidth would be the full probability to escape from the attracting resonance region of the stable orbit, and the calculation of this is beyond the scope of the present paper, but since the damping of the oscillation is causing the decay, one would expect a number of the order of this damping for the inverse of the linewidth, which is again a good agreement, since we found 1.5 times the correct quantum result.

Let us briefly consider the case of the $Li^+$ ion: The frequencies of Eq. (30) can be easily recalculated for the case of $Li^+$ by using the material of Secs. III and IV with $Z=3$ and $\phi=1/22$. We find

$$\omega_1 \equiv \sqrt{\frac{3}{11}} \simeq 0.5222\cdots,$$

$$\omega_2 \equiv \sqrt{\frac{5+\sqrt{60}}{44}} \simeq 0.5382\cdots, \tag{34}$$

$$\omega_3 \equiv \sqrt{\frac{12}{7}}|\beta| \simeq 1.2792|\beta|,$$

and a simple resonance condition like

$$\omega_1 - \omega_2 + \omega_3 = 0 \tag{35}$$

will determine the value of $|\beta|$ to be

$$|\beta| = 0.0125.$$

The stable orbit has a Coulombian energy of

$$E = -2.932 \ \text{a.u.}$$

and the frequency of the $z$-oscillation mode is

$$w_z = 1.97 \ \text{a.u.},$$

again in very good agreement with quantum mechanics. The first two quantum energies of para-lithium $Li^+$ [35] are ground state, $1^1S{:}E=-7.278$ a.u. and $2^1P{:}E=-5.30$ a.u. The frequency of the dipole transition becomes $w=7.278-5.3=1.978$ a.u., in very good agreement with our above value of $w_z$ (within 1%). Notice that this time we started the resonance condition with $n=1$ instead of $n=2$ as for helium, and this is a so far mysterious selection rule. This is another place where our heuristic criterion is incomplete. We could have tried to obtain this information from the sign of the next term in the normal form (as discussed in Appendix D). We should get back to this heuristic criterion when more information is known about exact numerical results of electrodynamics with retardation for the two-electron atom.

To summarize this section, the heuristic criterion can lead us to a surprisingly good agreement for energies, frequencies of sharp lines, and linewidths. The frequencies obtained agree better than the energies and some selection rules for the stability are missing.

## VII. DISCUSSION AND CONCLUSION

Historically (1912), Coulombian many-electron atoms (''saturnian atoms'') were investigated by astronomers [32] prior to Bohr, who was well aware of these studies. For oscillations perpendicular to the plane of the orbit, the Coulomb dynamics is stable and the *ratio* of many lines obtained by Nicholson for perpendicular oscillations agreed with the spectra of the Orion nebula and the Solar corona [32]. Of course Nicholson was assuming a special radius for the orbits, which he did not know how to calculate, and this radius would disappear when one took the ratio of two lines of the stable manifold of an orbit. This is a manifestation of the scale invariance of the Coulomb dynamics (namely: all the frequencies in the tangent dynamics are pure numbers times the orbit's frequency). Bohr was originally favorable to the use of ordinary mechanics to describe those stable perpendicular oscillations [28]. For oscillations along the plane of the orbit, Nicholson first found the now well-studied Coulombian radial instability [2,23], which was then a hindrance for the theory [28]. It is of historical interest to stress that it was the radial instability that first motivated Bohr to postulate a discrete set of special, more stable orbits, which proved to be a very fruitful intuition [28]. The original critic of Bohr [28] to Nicholson [32], was that the circular instability would make the atoms too ''fragile'' to disintegration, and unable to emit a sharp line. This led Bohr to conjecture that, along some special stable orbits, some ''quantum'' mechanism would supersede normal mechanics and prevent the radial instability [28].

As regards where the circular orbits ultimately decay to, we conjecture that the lowest energy bound state in helium could be a ''nonmechanical'' orbit of symmetric collinear motion. This orbit has zero angular momentum and zero electric dipole moment. The avoided three-body collision can be provided by a singular mechanism analogous to that of Eliezer's theorem for hydrogen [30,31]. In isolated atomic hydrogen with self-interaction, Eliezer's theorem predicts that the electron will never fall onto the proton. This counterintuitive result is closely related to the fact that one has always dipole radiation in atomic hydrogen. The symmetric colinear motion in helium is not radiating in dipole, and therefore one could expect a physical solution, differently from the case of hydrogen [31]. It is intersting to compare these possible zero-angular-momentum singular solutions to the divergent series of the Lamb shift [19,20] in quantum mechanics. The Lamb shift appears in quantum mechanics because of a singular interaction with the electromagnetic field and is only pronounced for states of zero angular momentum [19,20]. In the dynamical approach, the solutions with zero angular momentum and zero dipole will be nonmechanical colinear orbits that get very close to the nucleus. This in turn causes the Page series to diverge or to be asymptotic at the best (by analogy, one should call this the ''dynamical Lamb shift'').

A recent use of resonances in perturbation theory worth mentioning here was on the problem of the time of stability of integrable tori. Here one exploits the resonances among the unperturbed frequencies [39,43]. Those results go under the name of Nekhoroshev bounds for Arnold Diffusion. Simply stated, the results say that the actions of an $\epsilon$-perturbed Hamiltonian system are kept approximately constant for a time of the order of

$$T \simeq (1/\epsilon)\exp([1/\epsilon]^a),$$

where $a \equiv 1/d$ and $d$ is the maximal number of unperturbed frequencies linearly independent over the rationals [29,41]. Every time there is a resonance among the frequencies, $d$ is reduced and the torus has an exponentially longer time of stability. This phenomenon is named ''stability by resonance'' [41]. In connection with these modern Arnold diffusion results, it is interesting to mention that they shone new light on the old problem of the ultraviolet catastrophe, which was the historical motivation for Plank's hypothesis [29,42]. Today, also many numerical results exist showing that a set of coupled oscillators might never reach equipartition [43], the reason being a superslow Arnold diffusion.

As regards further research to be done, we have only been able to study the circular stationary orbits, and it would be of much interest to study the linear stability of general elliptical stationary orbits. Along elliptical orbits one can still use regular perturbation theory, in the same way used for circular orbits here, but now the parametric problem associated is much more complicated. It might be that the use of the Kustanheimo transformation [44,45] to regularized coordinates will simplify the problem of elliptical orbits. For zero-angular-momentum collinear orbits, it might be necessary to introduce the retardation effects in a nonperturbative way. (Because the page series will divenge.)

One would expect some discussion about spin: Notice that our simplified dynamical system is based on classical point-like charges with no spin. Of course, we could include spin in the dynamics in a phenomenological way, similar to the usual way it is introduced in quantum mechanics. We have not done that yet. Second, quantum mechanical spin is a relativistic effect and introduces a correction comparable to the second-order retardation correction [19,20]. As a matter of fact, the Breit operator for helium is actually produced starting from the Darwin Lagrangian [19,20]. We believe that a correct discussion of spin issues can only be made after we know more details about our dynamics, such as the frequency correction due to motion along resonance islands and etc.

As a summary, we presented a complete account of the linear stability of a two-electron atom along circular orbits in the presence of retardation and self-interaction. We calculated all the linear eigenvalues up to third order in $v/c$. We considered the necessary condition for a resonant constant and showed how to construct this extra analytic constant. From the condition of the resonant normal form we derived a heuristic criterion to determine the energy of the surviving resonant orbits. We also found that the frequency of the sharp lines in the tangent dynamics of stable orbits agrees well with the frequencies in the spectrum of helium. We do not know of prior results on the existence of these stable electromagnetic orbits, which appear as a genuine effect of the nonlinear dynamics prescribed by Maxwell's electrodynamics to atomic physics. We have barely touched the study of this dynamical system, and much research remains to be done.

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## APPENDIX A: FLOQUET ANALYSIS

The linearization of the Coulombian equations of motion produces a time-dependent linear equation with coefficients periodic in time with period $2\omega$. If the Floquet exponents are all nondegenerate, one can find a complete set of solutions to the linearized dynamical equations in the form [36–38]

$$\begin{aligned}\delta\mathbf{x}_\alpha &= \exp(2i\omega\mu t)\sum_n \mathbf{x}_n^\alpha \exp(2in\omega t),\\ \delta\mathbf{x}_1 &= \exp(2i\omega\mu t)\sum_n \mathbf{x}_n^1 \exp(2in\omega t),\\ \delta\mathbf{x}_2 &= \exp(2i\omega\mu t)\sum_n \mathbf{x}_n^2 \exp(2in\omega t),\end{aligned} \tag{A1}$$

where the Floquet exponent $\mu$ is a complex number defined in the first Brillouin zone, $-1/2 < \mathrm{Re}(\mu) < 1/2$. From now on, an upper index should not be confused with an exponent, and takes values $\alpha$, 1, and 2 to label the nucleus, electron 1, and electron 2, respectively. Notice that for the Floquet components we use an upper index, but to label coordinates as functions of time, as in Eq. (7), we use a lower index (to distinguish it from the Floquet components). To bring the variational equations to normal form, we define the coordinates

$$\xi_n^\kappa \equiv \frac{1}{\sqrt{2}}(x_n^\kappa - iy_n^\kappa), \quad \chi_n^\kappa \equiv \frac{1}{i\sqrt{2}}(x_n^\kappa + iy_n^\kappa),$$

where again the upper index is not to be confused with an exponent, and takes the values $\alpha$, 1, and 2. Next we calculate Hill's secular determinant [36], which reduces to the evaluation of a $6\times 6$ determinant in this case of circular orbits. As a simplification, let us define $\overline{n} \equiv n+\mu$ to be the running variable in the summations of Eq. (A1). Notice that $\overline{n}$ defines a frequency of linear oscillation in units of $2\omega$, as of Eq. (A1). Last, to introduce the physical intuition in the problem and explore the symmetries, it is convenient to define the coordinates $\mathbf{x}_r$ and $\mathbf{x}_d$ as

$$\begin{aligned}\mathbf{x}_r &\equiv \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_\alpha,\\ \mathbf{x}_d &\equiv \mathbf{x}_1 - \mathbf{x}_2,\end{aligned} \tag{A2}$$

respectively, which introduces two new letters, $r$ and $d$, to appear as labels. In helium, $(Z=2)$, the coordinate $\delta\mathbf{x}_r$ is the dipole moment, and a nonzero amplitude for this variation will always mean that the normal mode radiates in dipole. The dynamics of the $\delta\mathbf{x}_d$ variation is decoupled from the $\delta\mathbf{x}_\alpha$ and $\delta\mathbf{x}_r$ variations about circular orbits. Before we write the equations, we need still another definition

$$U_n^\kappa \equiv \frac{1}{\sqrt{2}}\xi_{n-1}^\kappa + \frac{i}{\sqrt{2}}\chi_{n+1}^\kappa,$$
$$V_n^\kappa \equiv \frac{-i}{\sqrt{2}}\xi_{n-1}^\kappa - \frac{1}{\sqrt{2}}\chi_{n+1}^\kappa. \tag{A3}$$

It is convenient to think of the quantities $U_n^\kappa$ and $V_n^\kappa$ as the $x$ and $y$ components of a two-dimensional vector $\mathbf{K}_n^\kappa$. At this point one should take a look at Appendix B, where we write the variation of the useful functional forms in terms of the vector Floquet components. Let us start by considering variations along the plane of the orbit. Using the results of Appendix B, the planar variational equations of Eq. (7) can be written most simply in terms of the Floquet components of Eq. (A1) as

$$-4\overline{n}^2\mathbf{x}_n^d - \frac{1}{2}(\mathbf{x}_n^d + 3\mathbf{K}_n^d) = \mathbf{0},$$
$$-4\overline{n}^2\mathbf{x}_n^\alpha + 4Z\phi(\mathbf{x}_n^r + 3\mathbf{K}_n^r)/\varrho = \mathbf{0}, \tag{A4}$$
$$-4\overline{n}^2\mathbf{x}_n^r - 4Z\phi\left(1+\frac{2}{\varrho}\right)(\mathbf{x}_n^r + 3\mathbf{K}_n^r) = \mathbf{0},$$

where $\varrho$ is defined as the ratio of the mass of the nucleus to the electronic mass. For example, in the case of helium $\varrho \approx 7344$. For the Coulomb dynamics only, also the last equation of (A4) depends only on the radiation coordinate. It is good to have a scheme in mind to keep track of what we have done so far: Eq. (A4) is the variation of Eq. (7) divided by $M_e\omega_0^2$, which we write schematically as

$$\frac{M\,\delta\mathbf{A}^{(0)} - \delta\mathbf{F}^{(0)}}{M_e\omega_0^2} = \mathbf{0}.$$

In Sec. IV we consider the variation of the relativistic corrections to the acceleration, which we call $\delta M\mathbf{A}^{(2)}$, and the Page series second order force, which we call $\delta\mathbf{F}^{(2)}$. The equations of motion will then be written schematically as

$$\frac{M\,\delta\mathbf{A}^{(0)} - \delta\mathbf{F}^{(0)}}{M_e\omega_0^2} + \frac{\delta M\mathbf{A}^{(2)} - \delta\mathbf{F}^{(2)}}{M_e\omega_0^2} + \cdots = \mathbf{0}.$$

In this way, we keep adding higher-order matrices and calculating Hill's secular determinant up to an order. Since we will be adding the matrices, it is necessary to keep some convention about the order in which the equations appear. The convention is that we always repeat what we did in Eq. (A4), that is, first the equation of motion for $\delta\mathbf{x}_d$ divided by $M_e\omega_0^2$, then the equation of motion for the $\delta\mathbf{x}_\alpha$ divided by $M_\alpha\omega_0^2$ and then the equation for $\delta\mathbf{x}^r$ divided by $M_e\omega_0^2$.

We recall that $\overline{n} = n + \mu$, and there is one equation for every value of $n$. In principle this would lead to an infinite Hill's secular determinant, which is the case for an elliptic periodic orbit. For the case of circular orbits, when we go to the $\xi$ and $\chi$ variables, we find that the $\xi_n$ variables couple only to $\chi_{n+1}$ and vice versa. This is obtained by taking linear combinations of the $x$ and $y$ components of the vectorial equations (A4), a procedure that we call ''unvectorizing.'' We developed this convenient mnemonic method of sketching the calculation to avoid making mistakes in an otherwise lengthy algebra (even though we did it with the symbolic manipulator Maple).

## APPENDIX B: VARIATION OF THE COULOMB INTERACTION

In this Appendix we calculate the variation of the terms appearing in the Page series, Eq. (2). To obtain the variation of the Coulomb force along the plane of the orbit, one has to evaluate

$$\delta\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \frac{\delta\mathbf{x}}{r^3} - \frac{3(x\,\delta x + y\,\delta y)\mathbf{x}}{r^5}, \tag{B1}$$

where $r = |\mathbf{x}|$ and $(\delta x, \delta y)$ are the functions of time representing the variation about the circular orbit

$$x = \pm r\cos(\omega t), \quad y = \pm r\sin(\omega t). \tag{B2}$$

Substituting Eq. (B2) into Eq. (B1) we obtain

$$\delta\left(\frac{x}{r^3}\right)_n = \{\delta x - \tfrac{3}{2}[1+\cos(2\omega t)]\delta x - \tfrac{3}{2}\sin(2\omega t)\,\delta y\}_n,$$
$$\delta\left(\frac{y}{r^3}\right)_n = \{\delta y - \tfrac{3}{2}[1-\cos(2\omega t)]\delta y - \tfrac{3}{2}\sin(2\omega t)\,\delta x\}_n, \tag{B3}$$

where the subscript $n$ indicates the $n$th Floquet component. Using Eq. (A1) we find $(\delta x)_n = x_n$ and $(\delta y)_n = y_n$ and

$$\delta\left(\frac{x}{r^3}\right)_n = -\frac{1}{2r^3}(x_n + 3U_n),$$
$$\delta\left(\frac{y}{r^3}\right)_n = -\frac{1}{2r^3}(x_n + 3V_n), \tag{B4}$$

where we have used $U_n$ and $V_n$ as defined by Eq. (A3). An economic way to write this equation is

$$\delta\left(\frac{\mathbf{x}}{r^3}\right)_n = -\frac{1}{2r^3}(\mathbf{x}_n + 3\mathbf{K}_n).$$

When we consider the second-order terms of the Page series and the relativistic correction of the mass, we need to evaluate the variation of many other functions besides $x/r^3$. This is done in Appendix C, in a way analogous to the above calculations. Last, along the $z$ direction the variation is simply

$$\delta\left(\frac{z}{r^3}\right) = \frac{\delta z}{r^3}.$$

## APPENDIX C: VARIATION OF THE SECOND-ORDER FORCES

In this Appendix we consider the terms of order $(v/c)^2$ in the force. One contribution comes from the first relativistic correction to the electronic masses (the $\alpha$ particle is at rest). The $x$ component of this term is

$$\delta M_e a_x^{(2)} = \frac{M_e}{c^2}\,\delta(\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}})\dot{x}_e + \frac{M_e}{2c^2}\,\delta(|\mathbf{x}_e|^2\ddot{x}_e).$$

Substituting the equation for a circular orbit, we find

$$(\delta M_e a_x^{(2)}) = \frac{M_e\omega^2}{2}|\beta|^2\left[[2-\cos(2\omega t)]\frac{\delta\ddot{x}_e}{\omega^2} - \sin(2\omega t)\,\frac{\delta\ddot{y}_e}{\omega^2} + 2\left(\sin(2\omega t)\,\frac{\delta\dot{x}_e}{\omega} - \cos(2\omega t)\,\frac{\delta\dot{y}_e}{\omega}\right)\right].$$

Using the above equation, equation (A1) for $\delta\mathbf{x}$ and definition (A3) for $U_n$ and $L_n$ we obtain the Floquet component

$$(\delta M_e a_x^{(2)})_n = M_e\omega^2|\beta_e|^2(-4\overline{n^2}x_n^e + 2\overline{n^2}U_n^e + 2in\overline{V}^e).$$

The $y$ component can be calculated in an analogous way. To write the two components in a concise vector form we define an extra useful vector quantity

$$\tilde{\mathbf{K}}_n^e = \begin{bmatrix} V_n \\ -U_n \end{bmatrix}.$$

This quantity is obtained from the vector $\mathbf{K}$ [defined just below Eq. (A3) by exchanging the components and changing the sign of the second component]. Making use of the above, we write the relativistic second-order force as

$$(\delta M_e \mathbf{a}^{(2)})_n = M_e\omega^2|\beta_e|^2(-4\overline{n^2}\mathbf{x}_n^e + 2\overline{n^2}\mathbf{K}_n^e + 2i\overline{n}\tilde{\mathbf{K}}^e).$$

Using a procedure analogous to the above, we can obtain the variation of the second-order electric interaction of Eq. (2). First, the variation of the second-order electric field produced by the electron at the point $p$ on the observation point $o$ is calculated. The vector pointing from the electron to the point $p$ (where another particle is) is $\mathbf{x} = \mathbf{x}_k - \mathbf{x}_e$ ($\kappa$ designates the other particle). In all cases of interest we have $\mathbf{x} = -g\mathbf{x}_e$. It is easy to verify that for electron-proton interaction $g=1$ and for electron-electron interaction $g=2$. According to Eq. (2), the variation of the electric field is proportional to the product of the charges times

$$\delta\left(\frac{|\beta_e|^2\mathbf{x}}{2r^3} - \frac{\dot{\boldsymbol{\beta}}_e}{2rc} - \frac{(\mathbf{n}\cdot\dot{\beta}_e)\mathbf{n}}{2rc^2}\right)_n = \frac{|\beta_e|^2}{2r^3}\left\{\left(\frac{g-1}{2}\right)(\mathbf{x}_n + 3\mathbf{K}_n) + 2\overline{n^2}g^2(\mathbf{x}_n^e + \mathbf{K}_n^e) + 4\overline{n^2}\left(\frac{r}{R}\right)^2\mathbf{x}_n^e - 2ig\overline{n}(\tilde{\mathbf{x}}_n^e - \tilde{\mathbf{K}}_n^e)\right\}, \tag{C1}$$

where $r$ is the interparticle distance and $R$ the radius of the electronic orbits. The tilde over $\mathbf{x}$ means the vector obtained from $\mathbf{x}$ by exchanging the components and changing the sign of the second component.

$$\tilde{\mathbf{x}}_n^\kappa = \begin{bmatrix} y_n^\kappa \\ -x_n^\kappa \end{bmatrix},$$

where again the superscript $\kappa$ can take the values 1, 2, $\alpha$, $r$, and $d$, to represent a particle's coordinate or a combination of the coordinates, as defined in Eq. (A2). The second-order electric field produced by the nucleus on each electron is

$$-\frac{2e^2|\beta|^2\overline{n^2}}{R^3}(3\mathbf{x}_n^\alpha + \mathbf{K}_n^\alpha),$$

the same for the two electrons. Last, we calculate the variation of the second-order magnetic forces. The $\alpha$ particle is at rest and does not produce any magnetic field. The two electrons produce a magnetic field at the center of the orbit, and the Lorentz magnetic force over the nucleus is

$$(\delta\mathbf{F}_M^\alpha)_n = -\frac{4Zie^2\omega l_z}{M_e c^2 R^3}\overline{n}\tilde{\mathbf{x}}_n^\alpha,$$

where $l_z$ is the $z$ component of the angular momentum of one electron. The variation of the electron-electron magnetic force done by electron 2 over electron 1 is

$$(\delta\mathbf{F}_{21})_n = \frac{e^2|\beta_e|^2}{r^3}\{2i\overline{n}(2\tilde{\mathbf{x}}_n^{e1} - \tilde{\mathbf{x}}_n^{e2} + \tilde{\mathbf{K}}_n^{e2}) - (\mathbf{x}_n^d + \mathbf{K}_n^d)\}. \tag{C2}$$

For the force done by electron 1 over electron 2 just interchange the indices and the sign in front of $\mathbf{x}_n^d$ and $\mathbf{K}_n^d$.

## APPENDIX D: CONSTRUCTION OF THE EXTRA COMPLEX CONSTANT OF MOTION

To construct the resonant normal form, it is again convenient to move to the rotating system. For this section we disregard the acceleration of the nucleus, so that $\mathbf{x}_\alpha = 0$. It is also convenient to use the Hamiltonian formalism associated with the Darwin Lagrangian, written in terms of the coordinates $\mathbf{x}_r$ and $\mathbf{x}_d$, as defined by Eq. (A2), and the conjugate momenta $\mathbf{p}_r$ and $\mathbf{p}_d$. We make a last transformation to the normal mode coordinates of the Jacobian matrix about the fixed point corresponding to the circular orbit. Let $u_i$, $i=1,\ldots,12$, be the coordinates corresponding to the normal modes of frequencies $\omega_i$, $i=1,\ldots,12$, respectively. These coordinates will be complex-valued linear functions of the coordinates and momenta. Because of the simplectic symmetry, the eigenvalues exist in pairs $\omega$ and $-\omega$ [40]. We order the modes such that $u_{12-i}$ is the coordinate corresponding to the frequency $\omega_{12-i} = -\omega_i$. Because there are six coordinates and six conjugate momenta, we have twelve normal coordinates. For example, the normal mode solution for the coordinate $u_1$ will be given by

$$u_1(t)=u_1(0)\exp(i\omega_1 t),$$

and so on for the other coordinates. We define these coordinates to describe the linearization about the fixed point corresponding to the circular orbit, so that at the circular orbit all coordinates are zero.

We want to find an extra constant with an analytic form, which implies it has a formal Taylor series about the fixed point (not necessarily convergent). For instance, the integer powers appearing in this Taylor series are the cause of the integer numbers in the necessary resonance condition [8]. Before we go on, let us introduce some definitions to simplify the exposition: Because we consider a Maclaurin expansion of the constant of motion, some notation concerning monomials is in place: We write a monomial in the coordinates $u_j$, $j=1,\ldots,12$ as

$$U^{\mathbf{k}}\equiv u_1^{k_1}u_2^{k_2}\cdots u_{12}^{k_{12}},$$

where $\mathbf{k}$ is a twelve-dimensional vector of integer components. For example, if the $i$th coordinate is not present in the monomial, then $k_i=0$. We use the convention that when $k_i$ is negative, the coordinate to be used is the complement of $k_i$ to 12 with the positive power $(k_{12-i}=-k_i)$. For example,

$$u_1(t)^{-1}\equiv u_7(t),$$

and so on for all the other coordinates. Mode 7, by definition, has a frequency that is the negative of $\omega_1$. With this convection, even though we are working with negative integers, all the powers of the coordinates in the series for the constant are positive, as it should for any good Taylor series.

Let us define the order of a monomial by

$$o=|k_1|+|k_2|+\cdots+|k_{12}|,$$

and the frequency associated with a monomial as

$$\Delta\omega_{\mathbf{k}}=k_1\omega_1+k_2\omega_2+\cdots+k_{12}\omega_{12}.$$

The necessary condition for the existence of the extra analytic constant is that some resonance condition must be satisfied among the linear eigenvalues. In the following we show how to construct the constant starting from a leading resonance [4–6,8]. Let us assume that a resonance defined by $k_0$ exists

$$\Delta\omega_{\mathbf{k_0}}=0. \tag{D1}$$

Our proposed analytic constant must have a formal Maclaurin series expansion given by

$$C=\sum_{\mathbf{k}} C_{\mathbf{k}}U^{\mathbf{k}},$$

where the $C_{\mathbf{k}}$ are constant numbers. This is an analytic function by construction, because it involves only integer powers of the coordinates. We now construct a constant where the lowest-order monomial is defined by $\mathbf{k}_0$ as of Eq. (D1), plus higher-order monomials necessary to vanish the time derivative at higher orders. For example, if the resonance condition (31) is our leading resonance, then we can construct a nontrivial analytic constant by starting the Maclaurin series with $k_1=1$, $k_8=1$, $k_3=2$, and $k_2=k_3=\cdots=k_{12}=0$, as of Eq. (31). To produce the successive powers of the constant, let us write the constant as

$$C=U^{\mathbf{k}_0}+\sum_{\mathbf{k}_1} C_{\mathbf{k}_1}U^{\mathbf{k}_1}+\cdots, \tag{D2}$$

where we have separated out the contribution of the leading term at $\mathbf{k}_0$, the contribution of the monomials of immediately next order, which we index with $\mathbf{k}_1$, and the dots represent monomials of higher than immediately next orders. This is a complex-valued function, which means two real functions of phase space. Of course, starting the series with the lattice vector $-\mathbf{k}_0$ will produce the complex conjugate of the same function. Next we evaluate the time derivative of Eq. (D2).

The time derivative of the leading monomial produces the same monomial multiplied by $\Delta\omega_{\mathbf{k}_0}=0$, plus higher-order terms, as

$$\frac{d}{dt}(U^{\mathbf{k}_0})=\Delta\omega_{\mathbf{k}_0}U^{\mathbf{k}_0}+\sum_{\mathbf{k}_1} N_{\mathbf{k}_1}U^{\mathbf{k}_1}+\cdots.$$

This monomial is then almost a constant, up to higher order monomials. To produce the higher monomials of the constant, let us focus on one such term of immediately next order, corresponding to a vector $\mathbf{k}_1$ and with coefficient $N_{\mathbf{k}_1}$. The time derivative of $C$ is

$$\frac{dC}{dt}=\Delta\omega_{\mathbf{k}_0}C_{\mathbf{k}_0}U^{\mathbf{k}_0}+\sum_{\mathbf{k}_1} N_{\mathbf{k}_1}U^{\mathbf{k}_1}+\cdots+\sum_{\mathbf{k}_1}\Delta\omega_{\mathbf{k}_1}C_{\mathbf{k}_1}U^{\mathbf{k}_1}+\cdots. \tag{D3}$$

It is easy to see that the next-order nonlinear contribution of the leading monomial to the time derivative can be canceled by choosing

$$C_{\mathbf{k}_1}=-N_{\mathbf{k}_1}/\Delta\omega_{\mathbf{k}_1}.$$

It is well known that this perturbation scheme does not produce a convergent constant, which is not a problem if we are investigating stability for a finite time scale. The usual procedure is to terminate the series using some optimal truncation [39]. This produces a quasiconstant for a long timescale, which still provides stability in the finite time for the special orbits. The detailed investigation of this issue is beyond the scope of the present paper. Here we are mainly concerned with exploring the necessary condition for the existence of such a constant.

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